Department of Mechanical Engineering
Faculty of Engineering and Architecture

***Closed book***
***Scientific calculators are allowed***
***Return the entire question booklet and other scratch sheets to the instructor***
***Show all your work for full credit and circle your answers***

April 11, 2012

**Duration: 90 minutes**

| Question | Grade |
|---|---|
| 1 | /30 |
| 2 | /20 |
| 3 | /30 |
| 4 | /20 |
| **Total** | |

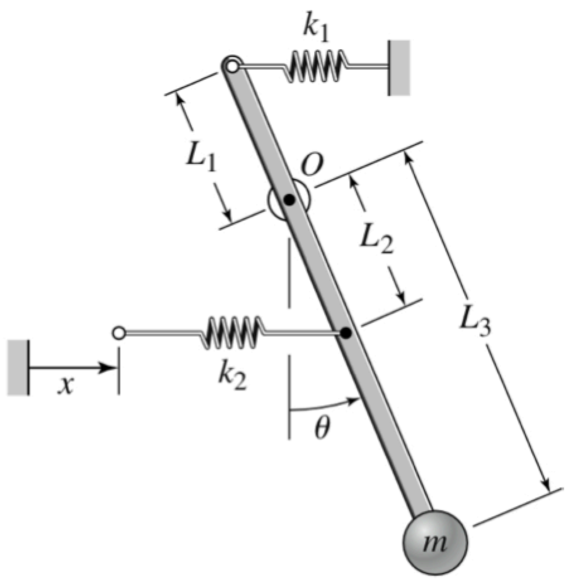
| | |
|---|---|
| Name | |
| Student ID | |

***Good luck***

## Problem #1: (30)

The mass $m$ is attached to a rigid lever having negligible mass and negligible friction in the pivot. The input is the displacement $x$. When $x$ and $\theta$ are $0$, the springs are at their free length. Assuming that $\theta$ is small, solve the following:

a) The free body diagram of the lever
b) The equations of motion for $\theta$ with $x$ as the input
c) The equations of motion in terms of the appropriate state variables




**a) FBD**

$CCW : \theta$

$CW : J\ddot{\theta} = mL_3^2\ddot{\theta}$

$CW : L_1 f_{k_1} = L_1 k_1 \left(L_1\theta\right)$

$CW : L_2 f_{k_2} = L_2 k_2 \left(L_2\theta - x\right)$

$CW : mgL_3 \sin\theta = mgL_3\theta$

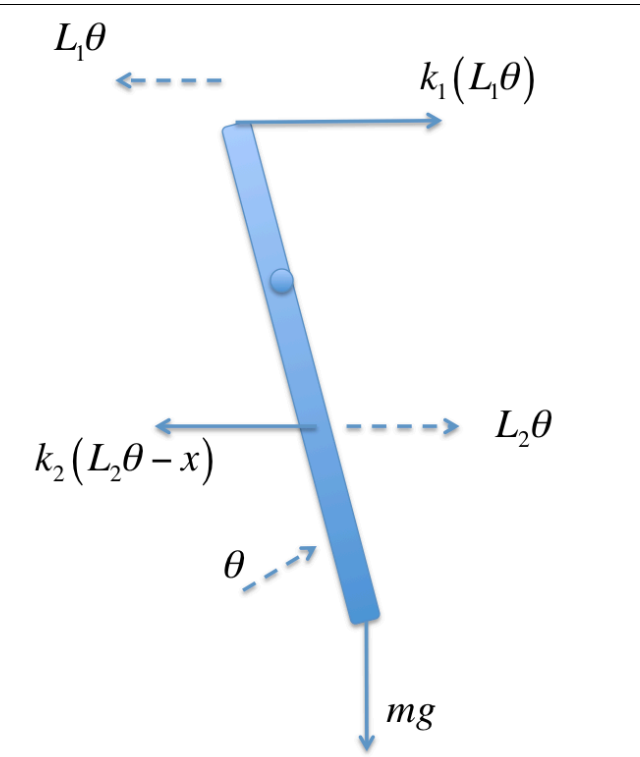
**b) EOM**

$$mL_3^2\ddot{\theta} + L_1k_1\left(L_1\theta\right) + L_2k_2\left(L_2\theta - x\right) + mgL_3\theta = 0$$

$$\Rightarrow mL_3^2\ddot{\theta} + \left(k_1L_1^2 + k_2L_2^2 + mgL_3\right)\theta = k_2L_2x$$

**c) State variables**

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{1}{mL_3^2}\left(k_2L_2x - \left(k_1L_1^2 + k_2L_2^2 + mgL_3\right)\theta\right) \end{pmatrix}$$



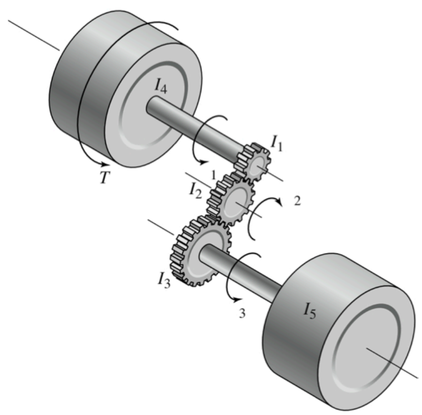
## Problem #2: (20)

For the geared system shown below, assume that the shaft inertias and gear inertias, $I_1$, $I_2$, and $I_3$ are negligible. The motor and load inertias are $I_4$ and $I_5$, respectively. The speed ratios are

$$\frac{\omega_1}{\omega_2} = \frac{\omega_2}{\omega_3} = N$$

Derive the following:

a) The free body diagrams
b) The system model in terms of the speed $\omega_3$, with the applied torque $T$ as the input

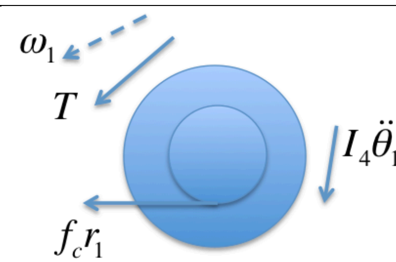


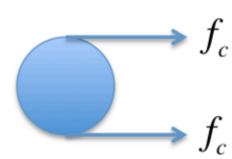

$$FBD: I_4$$
$$CW: I_4\dot{\omega}_1$$
$$CCW: T$$
$$CW: f_c r_1$$
$$\Rightarrow I_4\dot{\omega}_1 = T - f_c r_1$$

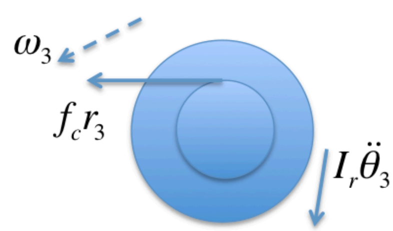
$$FBD: I_5$$
$$CW: I_5\dot{\omega}_3$$
$$CCW: f_c r_3$$
$$I_5\dot{\omega}_3 = f_c r_3$$

$$\frac{r_3}{r_1} = \frac{\omega_1}{\omega_3} = \frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3} = N^2$$

$$f_c = \frac{1}{r_1}\left(I_4\dot{\omega}_1 - T\right)$$
$$\Rightarrow I_5\dot{\omega}_3 = -f_c r_3 = \frac{r_3}{r_1}\left(T - I_4\dot{\omega}_1\right)$$
$$\Rightarrow \left(I_5 + N^4 I_4\right)\dot{\omega}_3 = N^2 T$$



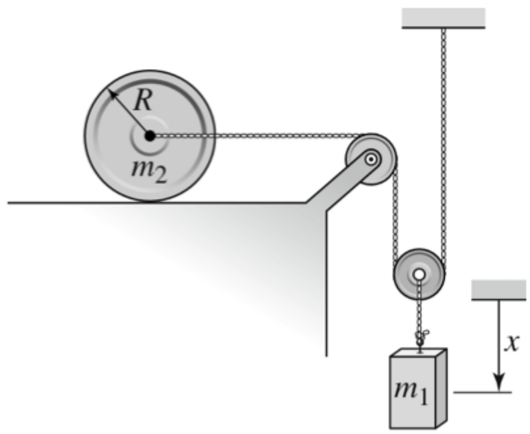



## Problem #3: (30)

Assume the cylinder below rolls without slipping. Neglecting the mass of the pulleys and derive the following:

a) The free body diagrams
b) The equation of motion of the system in terms of the displacement $x$
c) The equations of motion in terms of the appropriate state variables

*Hint: Do a summation of moments about the point of contact between the cylinder and the table.*




$FBD: m_1$

$m_1\ddot{x} = m_1 g - 2T$

$FBD: m_2$ : Summation of moments about the contact point

$$\left(\frac{1}{2}m_2R^2 + m_2R^2\right)\ddot{\theta} = TR$$

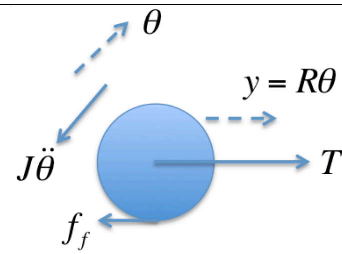
$$y = R\theta = 2x$$

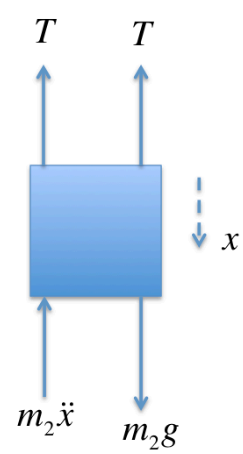
$$T = \frac{3}{2}m_2R\ddot{\theta}$$

$$\Rightarrow m_1\ddot{x} = m_1g - 3m_2R\ddot{\theta} = m_1g - 3m_2R\left(\frac{2\ddot{x}}{R}\right)$$

$$\Rightarrow (m_1 + 6m_2)\ddot{x} = m_1g$$

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{m_1}{m_1 + 6m_2}g \end{pmatrix}$$

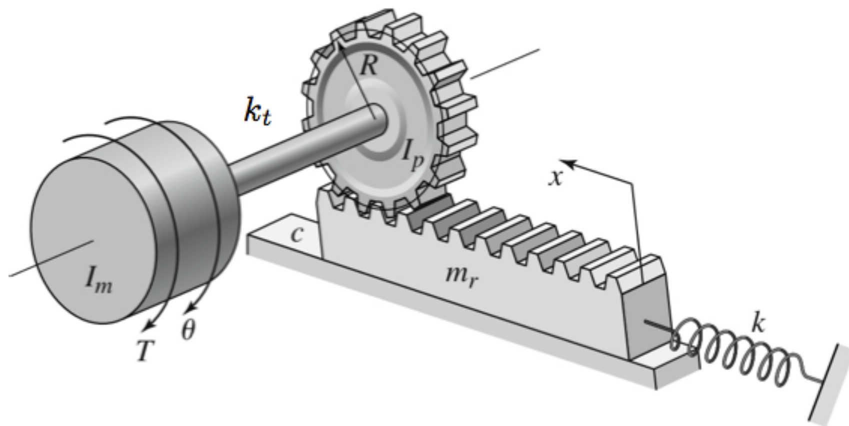


## Problem #4: (20)

Given a motor with inertia $I_m$ with a drive torque $T$ that is connected to a pinion with inertia $I_p$ and radius $R$. The shaft connecting the to motor to the pinion has a stiffness of $k_t$. The pinion is driving a rack whose mass is $m_r$. The rack has a spring attached to it and is fighting a viscous friction with coefficient $c$ as shown below.

Derive the governing equation or system of equations of the system below.




$FBD: I_m$

$CCW: I_m\ddot{\theta}$

$CCW: k_t(\theta - \theta_A)$

$CW: T$

$FBD: Gear$

$CCW: I_p\ddot{\theta}_A$

$CCW: k_t(\theta_A - \theta)$

$CW: f_c R$

$FBD: Rack$

$\rightarrow: m_r\ddot{x}$

$\rightarrow: kx$

$\rightarrow: c\dot{x}$

$\rightarrow: f_c$

$$\left.\begin{array}{l} I_m\ddot{\theta} + k_t(\theta - \theta_A) = T \\ I_p\ddot{\theta}_A + k_t(\theta_A - \theta) - f_c R = 0 \\ m_r\ddot{x} + c\dot{x} + kx = -f_c \\ \theta_A R = x \end{array}\right\} \Rightarrow \left\{\begin{array}{l} I_m\ddot{\theta} + k_t\left(\theta - \dfrac{x}{R}\right) = T \\ m_r\ddot{x} + c\dot{x} + kx = -\dfrac{k_t}{R}\left(\dfrac{x}{R} - \theta\right) - \dfrac{I_p}{R}\dfrac{\ddot{x}}{R} \end{array}\right.$$